Consider a semiconductor material in which the electronic band structure of the conduction band is given by the following relationship:

\[ E(k) = \sqrt{E_0^2 + \Delta^2 k^2} \]

(a) Compute the electronic density of states for the conduction band of this material in three dimensions, as a function of \( E_0 \) and \( \Delta \).

(b) Compute the electronic density of states for the conduction band of this material in two dimensions, assuming that confinement via infinite potential energy barriers in the third dimension yields a single confined state with energy \( E_1 \). Note that \( E_1 \) is the actual energy of the confined state, so the energy increase due to quantum confinement is \( E_1 - E_0 \). Express your result as a function of (as needed) \( E, E_1, E_0 \) and \( \Delta \).

(c) Compute the electronic density of states for the conduction band of this material in one dimension, assuming that confinement via infinite potential energy barriers in the other two dimensions yields a single confined state with energy \( E_{11} \). Note that \( E_{11} \) is the actual energy of the confined state, so the energy increase due to quantum confinement is \( E_{11} - E_0 \). Express your result as a function of (as needed) \( E, E_{11}, E_0 \) and \( \Delta \).

(d) Compute the effective mass \( m^* \) of the conduction band at the Brillouin zone center (\( k=0 \)), where the effective mass is defined as

\[ m^* = \hbar^2 \left( \frac{\partial^2 E}{\partial k^2} \right)^{-1} \]

Compare the two-dimensional density of states computed in part (b) with the two-dimensional density of states that would be obtained for a parabolic band with band structure \( E(k) = E_0 + \hbar^2 k^2 / 2m^* \), where the effective mass \( m^* \) has the value you just calculated. Explain your result.
1. a) \( E(E) = \sqrt{E_0^2 + \delta^2 k^2} \)

Recall that in general, in 3 dimensions, the total number of states with energy up to \( E \) can be calculated by taking the volume in \( k \)-space occupied by those states, and dividing by the \( k \)-space volume occupied by each electronic state \( (\frac{2\pi}{L})^3 \) to every 2 electronic states in 3 dimensions, where \( L \) is the side over which periodic boundary conditions are applied. Then the density of states in 3D, \( D_{3D}(E) \), is obtained by differentiating wrt energy:

\[
L^3 N_{\text{states}}(E) = \frac{4\pi}{2} \frac{k^3}{L^3} = \frac{2\pi}{3} \frac{k^3}{L^3} = \frac{8\pi}{3} \frac{k^3}{L^3}
\Rightarrow N_{\text{states}}(E) = \frac{1}{3\pi} \frac{k^3}{L^3}
\]

\[
N_{\text{states}}(E) = \frac{1}{3\pi} \left( \frac{E^2 - E_0^2}{\Delta^2} \right)^{\frac{3}{2}}
\]

\[
D_{3D}(E) = \frac{dN_{\text{states}}}{dE} = \frac{1}{\pi} \left( \frac{E^2 - E_0^2}{\Delta^2} \right)^{\frac{1}{2}} \frac{E}{\Delta^2}
\quad \text{for } E > E_0
\]

b) In 2 dimensions, the \( k \)-space "volume" occupied by states with energy up to \( E \) (and energy \( > E_1 \)) is a disc rather than a sphere. The \( k \)-space volume occupied by each electronic state is then \( \frac{1}{2} \left( \frac{2\pi}{L} \right)^2 \). Thus:

\[
L^2 N_{\text{states}}(E) = \frac{\pi k^2}{2} \frac{L^2}{2\pi} = \frac{1}{2\pi} k^2 L^2 \Rightarrow N_{\text{states}}(E) = \frac{1}{2\pi} k^2 \Rightarrow k = \sqrt{\frac{E^2 - E_0^2}{\Delta^2}} \quad \text{for } E > E_0
\]

\[
N_{\text{states}}(E) = \frac{1}{2\pi} \left( \frac{E^2 - E_0^2}{\Delta^2} \right)
\]

\[
D_{2D}(E) = \frac{dN_{\text{states}}}{dE} = \frac{1}{\pi} \left( \frac{E^2 - E_0^2}{\Delta^2} \right)^{\frac{1}{2}} \frac{E}{\Delta^2}
\quad \text{for } E > E_1
\]
c) In one dimension, states occupy points on a line in the $k_x$-direction in $k$-space (for confinement in the $y,z$-directions). The $k$-space "volume" occupied by each electronic state is then $\frac{2}{\pi} \left(\frac{2\pi}{E} \right)$.

Thus

\[ N_{\text{states}}(E) = \frac{2k_x}{\frac{1}{2} \left(\frac{2\pi}{E} \right)} = \frac{2}{\pi} k_x \Rightarrow N_{\text{states}}(E) = \frac{2}{\pi} k_x. \]

\[ E = \sqrt{E_0^2 + \Delta^2 (k_x^2 + k_y^2)} = \sqrt{E_0^2 + \Delta^2 k_x^2} \]

\[ k_x = \sqrt{\frac{E^2 - E_0^2}{\Delta^2}} \Rightarrow N_{\text{states}}(E) = \frac{2}{\pi} \left(\frac{E^2 - E_0^2}{\Delta^2} \right)^{1/2} \]

\[ \mathcal{D}_{1d}(E) = \frac{dN_{\text{states}}}{dE} = \frac{2}{\pi} \left(\frac{E^2 - E_0^2}{\Delta^2} \right)^{1/2} \frac{E}{\Delta^2} \quad \text{for } E > E_0. \]

d) \[ m^* = \hbar^2 \left(\frac{2^2 E}{2\hbar^2} \right)^{-1} \quad \text{with} \quad E(k) = \sqrt{E^2_0 + \Delta^2 k_x^2} \]

\[ \frac{2E}{\hbar k} = \frac{1}{2} \left(\frac{E_0^2 + \Delta^2 k_x^2}{\hbar^2} \right)^{1/2} 2\Delta k_x = \Delta \left(\frac{E_0^2 + \Delta^2 k_x^2}{\hbar^2} \right)^{1/2} k_x \]

\[ \frac{2E}{\hbar k} = \Delta \left[ \frac{1}{2} \left(\frac{E_0^2 + \Delta^2 k_x^2}{\hbar^2} \right)^{3/2} \frac{2\Delta^2 k_x^2}{\hbar k_x} + \left(\frac{E_0^2 + \Delta^2 k_x^2}{\hbar^2} \right) \left(\frac{E_0^2 + \Delta^2 k_x^2}{\hbar^2} \right)^{1/2} \right] \Rightarrow \frac{2E}{\hbar k} \bigg|_{k=0} = \frac{\Delta}{E_0} \]

\[ \Rightarrow m^* = \frac{k_x E_0}{\Delta^2} \quad \text{at } k = 0. \]

For a parabolic band \[ E(k) = \frac{\hbar^2 k^2}{2m^*}, \] we have for the density of states:

\[ N_{\text{states}}(E) = \frac{1}{2\pi} k^2 = \frac{m^*}{\pi \hbar^2} (E - E_0) \text{ with } E > E_1. \]

\[ \Rightarrow \mathcal{D}_{1d}(E) = \frac{dN_{\text{states}}}{dE} = \frac{m^*}{\pi \hbar^2} \text{ in the parabolic band case, or } \mathcal{D}_{1d}(E) = \frac{E_0}{\pi \hbar^2}. \]

Graphically, the two cases therefore compare as follows:
Thus, we see that in the non-parabolic case, the density of states is higher in 2D, and linear in $E$, rather than independent of $E$. The dependence on $E$ arises specifically from the non-parabolicity — only a perfectly parabolic band structure yields $\rho_{2D}(E)$ independent of $E$.

To see why the density of states is larger in the non-parabolic case, we can sketch $E(k^2)$ for both cases:

\[
E = \sqrt{E_0^2 + 2\alpha k^2} \sim k \text{ for large } k.
\]

More values of $k$ are contained within a given energy interval in this non-parabolic case than in the parabolic case, leading to a larger density of states.