MS/CTS EXAM SOLUTION – FALL 2014

For a possibility non W.S.S. process the power spectral density is given by

$$P(\omega) = \int_{-\infty}^{\infty} \bar{C}(\tau) e^{-i\omega\tau} d\tau$$

with

$$\bar{C}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_X(t+\tau, t)dt$$

and $R_X(t+\tau,t) = E[X(t+\tau)X(t)]$.

In the case of $X(t) = \sum_{n=-\infty}^{\infty} \alpha_n g(t-nT_0)$

we have

$$R_X(t,s) = E[X(t)X(s)]$$

$$= E\Big[\sum_{n=-\infty}^{\infty}\alpha_n g(t-nT_0)\sum_{m=-\infty}^{\infty}\alpha_m g(s-mT_0)\Big]$$

$$= \sigma^2 \sum_{n=-\infty}^{\infty} g(t-nT_0)g(s-nT_0)$$

because $E[\alpha_n\alpha_m] = \begin{cases}\sigma^2, & n=m\\ 0, & n\neq m\end{cases}$

Clearly $R_X(t+T_0, s+T_0) = R_X(t,s)$ and the process is cyclostationary and

$$\bar{C}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sigma^2\sum_{n=-\infty}^{\infty} g(t+\tau-nT_0)g(t-nT_0)dt$$

Define $f(t) \equiv \sigma^2 \sum_{n=-\infty}^{\infty} g(t+\tau-nT_0)g(t-nT_0)$

and note that $f(t+T_0) = f(t)$

To see that $f(t)$ is integrable over one period we must show that

$$\int_0^{T_0} |f(t)|dt < \infty$$

Now $$\int_0^{T_0} |f(t)|dt = \int_0^{T_0} \sigma^2 \left|\sum_{n=-\infty}^{\infty} g(t+\tau-nT_0)\, g(t-nT_0)\right| dt$$

$$\leq \sigma^2 \sum_{n=-\infty}^{\infty} \int_0^{T_0} |g(t+\tau-nT_0)\, g(t-nT_0)| dt$$

$t-nT_0 \equiv s$

$$\leq \sigma^2 \sum_{n=-\infty}^{\infty} \int_{-nT_0}^{-nT_0+T_0} |g(\tau+s)\, g(s)|\, ds$$

$$\leq \sigma^2 \int_{-\infty}^{\infty} |g(\tau+s)\, g(s)|\, ds$$

note that $g(t) = e^{-|t|}$

$$\leq \sigma^2 \int_{-\infty}^{\infty} e^{-|\tau+s|}\, e^{-|s|}\, ds < \infty$$

Using the "HINT" with the periodicity and integrability of $f(t)$

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt = \frac{1}{T_0}\int_0^{T_0} f(t)dt$$

$$= \frac{\sigma^2}{T_0} \sum_{n=-\infty}^{\infty} \int_0^{T_0} g(t+\tau-nT_0)\, g(t-nT_0)dt$$

again $t-nT_0 = s$

$$= \frac{\sigma^2}{T_0} \sum_{n=-\infty}^{\infty} \int_{-nT_0}^{-nT_0+T_0} g(\tau+s)\, g(s)ds$$

$$= \frac{\sigma^2}{T_0} \int_{-\infty}^{\infty} g(\tau+s)\, g(s)\, ds$$

Thus $$\bar{C}(\tau) = \frac{\sigma^2}{T_0}\int_{-\infty}^{\infty} g(\tau+s)g(s)\,ds$$

and $$P(\omega) = \int_{-\infty}^{\infty}\bar{C}(\tau)e^{-i\omega\tau}\,d\tau = \frac{\sigma^2}{T_0}|G(i\omega)|^2$$

with $$G(i\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,dt = \frac{2}{1+\omega^2}$$

"From Fourier Transform Pairs"

$$\therefore \quad \boxed{P(\omega) = \frac{\sigma^2}{T_0}\frac{4}{(1+\omega^2)^2}}$$

Finally we must prove the "HINT"

Let N be the integer satisfying

$$NT_0 \le T < (N+1)T_0$$

Then
$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t) = \underbrace{\lim_{T\to\infty}\frac{1}{2T}\int_{-NT_0}^{NT_0} f(t)\,dt}_{I_1} + \underbrace{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{-NT_0} f(t)\,dt}_{I_2} + \underbrace{\lim_{T\to\infty}\frac{1}{2T}\int_{NT_0}^{T} f(t)\,dt}_{I_3}$$

$$I_1 = \lim_{T\to\infty}\frac{1}{2T}\left(2N\int_0^{T_0} f(t)\,dt\right)$$ because of periodicity of $f(t)$

but $\frac{2N}{2T} \xrightarrow[T\to\infty]{} 1$

and $$I_1 \xrightarrow[T\to\infty]{} \frac{1}{T_0}\int_0^{T_0} f(t)\,dt$$

$$I_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{-NT_0} f(t)\,dt$$

$$|I_2| \le \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{-NT_0} |f(t)|\,dt$$

$$\le \lim_{T\to\infty} \frac{1}{2T} \int_{-(N+1)T_0}^{-NT_0} |f(t)|\,dt$$

f(t) periodic

$$\le \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T_0} |f(t)|\,dt$$

$$\le \lim_{T\to\infty} \frac{1}{2T} M$$

$M = \int_0^{T_0} |f(t)|\,dt < \infty$ because $f(t)$ is integrable

$\therefore$

$$I_2 \xrightarrow[T\to\infty]{} 0$$

Similarly $I_2 \xrightarrow[T\to\infty]{} 0$ ∎