Solutions – M.S. Exam. Spring 2013

There are two types of incorrect decision

(a) Choose λ_2 when λ_1 is present

(b) Choose λ_2 when λ_2 is present

The probability of λ_1 and λ_2 are equal; i.e.

$$P(\lambda_1) = P(\lambda_2) = \tfrac{1}{2}$$

The probability of an incorrect decision (probability of error) is:

$$P_e = P(\lambda_1)P(N(T) \ge M|\lambda_1) + P(\lambda_2)P(N(T) < M|\lambda_2)$$

$$= \tfrac{1}{2} P(N_1(T) \ge M) + \tfrac{1}{2} P(N_2(T) < M)$$

(1) $$\boxed{P_e = \frac{1}{2}\sum_{k=M}^{\infty} \frac{(\lambda_1 T)^k}{k!} e^{-\lambda_1 T} + \frac{1}{2}\sum_{k=0}^{M-1} \frac{(\lambda_2 T)^k}{k!} e^{-\lambda_2 T}}$$

using $P(N(T)=k) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$

for the Poisson probabilities

equivalent expressions

$$P_e = \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{M-1} \frac{(\lambda_2 T)^k}{k!} e^{-\lambda_2 T} - \frac{1}{2}\sum_{k=0}^{M-1} \frac{(\lambda_1 T)^k}{k!} e^{-\lambda_1 T}$$

or

$$P_e = \frac{1}{2}\sum_{k=M}^{\infty} \frac{(\lambda_1 T)^k}{k!} e^{-\lambda_1 T} - \frac{1}{2}\sum_{k=M}^{\infty} \frac{(\lambda_2 T)^k}{k!} e^{-\lambda_2 T} + \frac{1}{2}$$

Minimize P_e relative to T

$$0 \overset{set}{=} \left.\frac{dP_e}{dT}\right|_{T=T_0}$$

$$\frac{dP_e}{dT} = \frac{1}{2}\sum_{k=M}^{\infty}\frac{k\lambda_1^k T^{k-1}}{(k)!}e^{-\lambda_1 T} - \frac{1}{2}\lambda_1\sum_{k=M}^{\infty}\frac{(\lambda_1 T)^k}{k!}e^{-\lambda_1 T}$$

$$+\frac{1}{2}\sum_{k=0}^{M-1}\frac{k\lambda_2^k T^{k-1}}{k!}e^{-\lambda_2 T} - \frac{1}{2}\lambda_2\sum_{k=0}^{M-1}\frac{(\lambda_2 T)^k}{k!}e^{-\lambda_2 T}$$

set $k-1=l$ set $k=l$

$$\frac{dP_e}{dT} = \frac{1}{2}\lambda_1\sum_{l=M-1}^{\infty}\frac{(\lambda_1 T)^l}{l!}e^{-\lambda_1 T} - \frac{1}{2}\lambda_1\sum_{l=M}^{\infty}\frac{(\lambda_1 T)^l}{l!}e^{-\lambda_1 T}$$

$$+\frac{1}{2}\lambda_2\sum_{l=0}^{M-2}\frac{(\lambda_2 T)^l}{l!}e^{-\lambda_2 T} - \frac{1}{2}\lambda_2\sum_{l=0}^{M-1}\frac{(\lambda_2 T)^l}{l!}e^{-\lambda_2 T}$$

$$= \frac{1}{2}\lambda_1\frac{(\lambda_1 T)^{M-1}}{(M-1)!}e^{-\lambda_1 T} - \frac{1}{2}\lambda_2\frac{(\lambda_2 T)^{M-1}}{(M-1)!}e^{-\lambda_2 T}$$

setting $\left.\frac{dP_e}{dT}\right|_{T=T_0} = 0$

$$\frac{1}{2}\lambda_1\frac{(\lambda_1 T_0)^{M-1}}{(M-1)!}e^{-\lambda_1 T_0} = \frac{1}{2}\lambda_2\frac{(\lambda_2 T_0)^{M-1}}{(M-1)!}e^{-\lambda_2 T_0}$$

The solution is

(2) $$\boxed{T_0 = M\frac{\ln(\lambda_2/\lambda_1)}{(\lambda_2-\lambda_1)} = M\frac{\ln\lambda_2 - \ln\lambda_1}{\lambda_2-\lambda_1}}$$