# SOLUTION CTS MS EXAM – SPRING 2014

$$X(t) = \begin{cases} 0, & 0 \\ \sum_{k=1}^{N(t)} A_k, & N(t) \geq 1 \end{cases}$$

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, \cdots$$

$E[X(t)]$

$$E[X(t)] = \sum_{n=0}^{\infty} P(N(t)=n) E[X(t) \mid N(t)=n]$$

$$= 0 \cdot P(N(t)=0) + \sum_{n=0}^{\infty} P(N(t)=n) E\left[\sum_{k=1}^{n} A_k\right]$$

$$= \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \sum_{k=1}^{n} E[A_k]$$

(note $E[A_k] = 0$) $\rightarrow = 0$

$$\boxed{E[X(t)] = 0}$$

$E[X(t) X(s)]$

$\boxed{s \leq t}$

$$X(t) X(s) = \begin{cases} 0, & N(s) = 0 \\ \sum_{k=1}^{N(s)} A_k \sum_{l=1}^{N(s)} A_l, & N(s) \geq 1; N(t) - N(s) = 0 \\ \left(\sum_{k=1}^{N(s)} A_k + \sum_{k=N(s)+1}^{N(t)} A_k\right)\left(\sum_{l=1}^{N(s)} A_l\right), & N(s) \geq 1; N(t) - N(s) \geq 1 \end{cases}$$

$$E[X(t)X(s)] = \begin{cases} 0, & N(s)=0 \\ E\left[\sum_{k=1}^{N(s)} A_k \sum_{l=1}^{N(s)} A_l\right], & N(s)\ge 1,\ N(t)-N(s)=0 \\ E\left[\sum_{k=1}^{N(s)} A_k \sum_{l=1}^{N(s)} A_l\right] + E\left[\sum_{k=N(s)+1}^{N(t)} A_k \sum_{l=1}^{N(s)} A_l\right] & N(s)\ge 1,\ N(t)-N(s)\ge 1 \end{cases}$$

Note that $\sum_{k=N(s)+1}^{N(t)} A_k$ and $\sum_{l=1}^{N(s)} A_l$) are independent (different $A_k$s) for $N(s)\ge 1$ and $N(t)-N(s)\ge 1$

$\therefore$ In all cases

$$R_X(t,s) = \begin{cases} E\left[\sum_{k=1}^{N(s)} A_k \sum_{l=1}^{N(s)} A_l\right], & N(s)\ge 1 \\ 0, & N(s)=0 \end{cases}$$

$$E\left[\sum_{k=1}^{N(s)} A_k \sum_{l=1}^{N(s)} A_l\right] = \sum_{n=1}^{\infty} P(N(s)=n)\, E\left[\sum_{k=1}^{n} A_k \sum_{l=1}^{n} A_l \,\middle|\, N(s)=n\right]$$

$$= \sum_{n=1}^{\infty} \frac{(\lambda s)^n}{n!} e^{-\lambda s} \underbrace{E\left[\sum_{k=1}^{n} A_k \sum_{l=1}^{n} A_l\right]}_{\sigma^2 n}$$

$A_k$s are i.i.d; zero mean variance $\sigma^2$

$$= \sigma^2 \sum_{n=1}^{\infty} n \frac{(\lambda s)^n}{n!} e^{-\lambda s}$$

$n-1=p$

$$= \lambda \sigma^2 s \sum_{p=0}^{\infty} \frac{(\lambda s)^p}{p!} e^{-\lambda s} = \sigma^2 \lambda s$$

similarly for $t<s$

$$\boxed{R_X(t,s) = \sigma^2\lambda \min[s,t]}$$

MMSE

$$\mathcal{E} := E\left[\left(X(t)-\hat{X}(t)\right)^2\right]$$

$$\hat{X} = K X(t-t_1),\ t_1>0$$

orthogonality

$$E\left[\left(X(t)-\hat{X}(t)\right)X(t-t_1)\right] = 0$$

$$E[X(t)X(t-t_1)] - K E[X(t-t_1)X(t-t_1)] = 0$$

$$R_X(t,t-t_1) - K R_X(t-t_1,t-t_1) = 0$$

$$\sigma^2\lambda(t-t_1) - K\sigma^2\lambda(t-t_1) = 0$$

$$\therefore \boxed{K=1}$$