Math Question (equal weight each part)

Part 1

(a) Find the inverse of

$$\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

Solution

The inverse may be found by creating a partitioned matrix

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right].$$

Now apply the following row operations in sucession $3R_1 - R_2 \to R_2$, $R_1 - \frac{1}{2}R_2 \to R_1$ and $\frac{1}{4}R_2 \to R_2$. This yields

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array}\right]$$

so that thie inverse of $\mathbf{M}$ is

$$\mathbf{M}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}.$$

(b) Use the inverse to find $\mathbf{x}$ if

$$\mathbf{Mx} = \mathbf{b}$$

where

$$\mathbf{b} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$

Solution

Multiplying both sides by the inverse we have

$$\begin{aligned} \mathbf{x} &= \mathbf{M}^{-1}\mathbf{b} \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 5 \end{bmatrix}. \end{aligned}$$

The solution may be verified by using back substitution.

Part 2

Find all solutions for $\mathbf{x}$ and λ to the following equation

$$\mathbf{N}\mathbf{x} = \lambda\mathbf{x}$$

where

$$N = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution

The characteristic equation is

$$(2 - \lambda)(3 - \lambda) - 2 = 0$$

or

$$\lambda^2 - 5\lambda + 4$$

which has roots $\lambda = 1$ and $\lambda = 4$. These are the eigenvalues.

The eigenvector v_1 for $\lambda = 1$ is given by

$$\begin{bmatrix} 2 - \lambda_1 & 2 \\ 1 & 3 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \to x_1 + 2y_1 = 0 \to v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The eigenvector v_2 for $\lambda = 4$ is given by

$$\begin{bmatrix} 2 - \lambda_2 & 2 \\ 1 & 3 - \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \to x_2 - y_2 = 0 \to v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Part 3

Find all steady-state solutions for $x(t)$ and λ to the following equation

$$Rx(t) = \lambda x(t)$$

where R is the derivative operator given by

$$R = \frac{d^2}{dt^2} - 4\frac{d}{dt} + 6$$

Solution From your differential equations class, let the form of the steady-state solution be given by $x(t) = e^{j\omega t}$ where ω is an arbitrary frequency. Substituting this form into the equation and evaluating the derivatives, we have

$$\left(-\omega^2 - j4\omega + 6\right) e^{j\omega t} = \lambda e^{j\omega t}$$

or

$$\lambda(\omega) = -\omega^2 - j4\omega + 6.$$

Thus the eigenvalue for any value of ω is determined by the transfer function for that value of ω. The corresponding eigenfunction is $e^{j\omega t}$. This solution emphasizes that the eigenfunctions of a linear shift-invariant system are the complex exponentials with the eigenvalue given by the transfer function evaluated at the frequency ω.