Spring 2011 Master's Exam Math Question Solutions

Part 1

(i) Solve the following differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 29y(t) = 87$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 6$.
(ii) Sketch the solution over the range of $0 < t < 5$.
(iii) Assuming this solution describes the behavior of a RLC circuit, describe the nature of the response and label the on the sketch the physically significant parts of the solution.

Solution

The characteristic function for the natural response is $x^2 + 4x + 29$ and has solutions $-2 \pm j5$. The natural response is then

$$y_n(t) = e^{-2t}\left[A\cos(5t) + B\sin(5t)\right].$$

The forced solution is given by $29y_f(t) = 87$ which yields $y_f(t) = 3$.

The solution is then of the form

$$\begin{aligned} y(t) &= y_n(t) + y_f(t) \\ &= 3 + e^{-2t}\left[A\cos(5t) + B\sin(5t)\right] \end{aligned}$$

The constants are determined applying the initial conditions. For $y(0) = 0$, $A = -3$. Taking the derivative to apply the second initial condition, we have

$$y'(t) = e^{-2t}(5B\cos(5t) + 15\sin(5t)) - 2e^{-2t}(B\sin(5t) - 3\cos(5t))$$

Using $y'(0) = 6$, we have

$$6 + 5B = 6$$

or $B = 0$.

The complete solution is

$$y(t) = 3\left[1 - e^{-2t}\cos(5t)\right].$$

The solution corresponds to an underdamped circuit. A plot of the solution with the relevant parts of the response including the period $T = 2\pi/\omega = 2\pi/5$ is below in the figure below.

Part 2

(i) Find the eigenvalues and eigenvectors of the following matrix

$$M = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$$

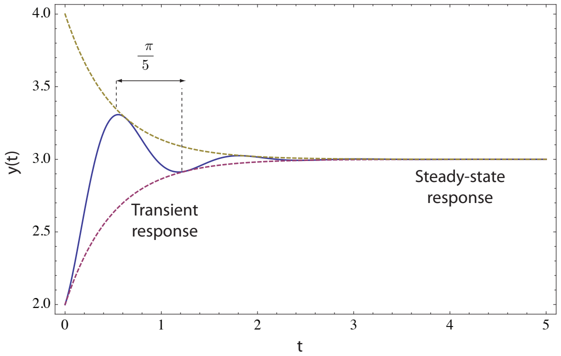

Figure 1: Sketch for Part 1

Solution
The eigenvalues are

$$\det\begin{bmatrix} 5-\lambda & 1 \\ -3 & 1-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 8 = 0$$
$$= (\lambda - 4)(\lambda - 2)$$

so that the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 4$.

The eigenvector v_1 is given by

$$\begin{bmatrix} 5-\lambda_1 & 1 \\ -3 & 1-\lambda_1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \rightarrow 3x_1 + 1y_1 = 0 \rightarrow v_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

The eigenvector v_2 is given by

$$\begin{bmatrix} 5-\lambda_2 & 1 \\ -3 & 1-\lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \rightarrow x_2 + y_2 = 0 \rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(ii) Find the matrix inverse M^{-1}

Solution

The inverse M^{-1} is determined using the following row operations $R_1 \rightarrow \frac{1}{5}R_1$, $R_2 \rightarrow R_2 + 3R_1$, $R_2 \rightarrow \frac{5}{8}R_2$ and $R_1 \rightarrow R_1 - \frac{1}{5}R_2$

$$\left[\begin{array}{cc|cc} 5 & 1 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{5} & \frac{1}{5} & 0 \\ -3 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & \frac{8}{5} & \frac{3}{5} & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{3}{8} & \frac{5}{8} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \underbrace{\begin{array}{cc} \frac{1}{8} & -\frac{1}{8} \\ \frac{3}{8} & \frac{5}{8} \end{array}}_{M^{-1}} \end{array}\right]$$

so that

$$M^{-1} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$