SOLUTION - CTS M.S. EXAM - FALL 2013

Let $h(t)$ be the impulse response corresponding to the transfer function $H(i\omega)$. Then

$$\hat{X}(t) = \int_{-\infty}^{\infty} h(\alpha)\left[X(t-\alpha)+N(t-\alpha)\right]d\alpha .$$

$$\mathcal{E} = E\left[(X(t)-\hat{X}(t))^2\right]$$

$$= E\left[X^2(t)\right] - 2E\left[X(t)\int_{-\infty}^{\infty} h(\alpha)\left[X(t-\alpha)+N(t-\alpha)\right]d\alpha\right]$$

$$+ E\left[\int_{-\infty}^{\infty} h(\alpha)d\alpha\int_{-\infty}^{\infty} h(\beta)d\beta\,\left(X(t-\alpha)+N(t-\alpha)\right)\left(X(t-\beta)+N(t-\beta)\right)\right]$$

Because $X(t)$ and $N(t)$ are independent and have zero mean

$$E\left[X(t)\left(X(t-\alpha)+N(t-\alpha)\right)\right] = R_X(\alpha)$$

$$E\left[\left(X(t-\alpha)+N(t-\alpha)\right)\left(X(t-\beta)+N(t-\beta)\right)\right] =$$

$$= R_X(\beta-\alpha) + R_N(\beta-\alpha)$$

So that

$$\mathcal{E} = R_X(0) - 2\int_{-\infty}^{\infty} h(\alpha)R_X(\alpha)d\alpha + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)\left[R_X(\beta-\alpha)+R_N(\beta-\alpha)\right]d\alpha\, d\beta$$

Expressing these correlations in terms of their power spectral densities

$$R_X(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, e^{i\omega\alpha}\, d\omega$$

$$R_X(\beta-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, e^{i\omega(\beta-\alpha)}\, d\omega$$

$$R_N(\beta-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_N(\omega)\, e^{i\omega(\beta-\alpha)}\, d\omega$$

$\therefore$

$$\mathcal{E} = R_X(0) - 2\,\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\,d\omega \int_{-\infty}^{\infty} h(\alpha)\, e^{i\omega\alpha}\, d\alpha$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\,d\omega \int_{-\infty}^{\infty} h(\alpha)\, e^{-i\omega\alpha}\, d\alpha \int_{-\infty}^{\infty} h(\beta)\, e^{i\omega\beta}\, d\beta$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} S_N(\omega)\,d\omega \int_{-\infty}^{\infty} h(\alpha)\, e^{-i\omega\alpha}\, d\alpha \int_{-\infty}^{\infty} h(\beta)\, e^{i\omega\beta}\, d\beta$$

But $\int_{-\infty}^{\infty} h(\alpha)\, e^{-i\omega\alpha}\, d\alpha = H(i\omega)$

$\therefore$

$$\mathcal{E} = R_X(0) - \frac{2}{2\pi}\int_{-\infty}^{\infty} H^*(i\omega)\, S_X(\omega)\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(i\omega)|^2 S_X(\omega)\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(i\omega)|^2 S_N(\omega)\, d\omega$$

But since $H(i\omega) = \begin{cases} 1\,; & -\Omega \le \omega \le \Omega \\ 0\,; & \text{otherwise} \end{cases}$

and the spectral densities are even functions of ω

$$\mathcal{E} = R_X(0) - \frac{2}{\pi}\int_0^{\Omega} S_X(\omega) + \frac{1}{\pi}\int_0^{\Omega} S_X(\omega)\,d\omega + \frac{1}{\pi}\int_0^{\Omega} S_N(\omega)$$

To find the value of Ω $(=\Omega_0)$ that minimizes $\mathcal{E}$

$$\left.\frac{d\mathcal{E}}{d\Omega}\right|_{\Omega=\Omega_0} \overset{\text{set}}{=} 0 = -\frac{1}{\pi}S_X(\Omega_0) + \frac{1}{\pi}S_N(\Omega_0)$$

The solution is that value of Ω_0 that satisfies $S_X(\Omega_0) = S_N(\Omega_0)$

From "Fourier Transform Pairs"

$$R_X(\tau) = e^{-a|\tau|} \longleftrightarrow S_X(\omega) = \frac{2a}{a^2+\omega^2}$$

$$R_N(\tau) = e^{-b|\tau|} \longleftrightarrow S_N(\omega) = \frac{2b}{b^2+\omega^2}$$

Solving for Ω_0

$$\frac{a}{a^2+\Omega_0^2} = \frac{b}{b^2+\Omega_0^2} \Rightarrow \boxed{\Omega_0 = \sqrt{ab}}$$

ANSWER

NOTE: This is not a problem to find the optimum filter. It is only required to find the bandwidth of a prescribed filter.