Statistical Signal Processing

Problem #1

(a) $$c_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n)\, x(n+|m|)$$

(b) $$I(f) = \sum_{m=-(N-1)}^{N-1} c_{xx}(m)\, e^{-j2\pi f m}$$

$$= \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi f n} \right|^2$$

(c) $$\hat{P}_{xx}(f) = \frac{1}{K} \sum_{k=0}^{K-1} \left\{ \frac{1}{LU} \left| \sum_{n=0}^{L-1} w(n)\, x_k(n)\, e^{-j2\pi f n} \right|^2 \right\}$$

where: L = segment length

$w(n)$ = window function (e.g. Hanning, Hamming, Kaiser-Bessel, etc.)

$x_k(n)$ = k^{th} segment where n is defined within the segment $0 \le n \le L-1$

$$U = \frac{1}{L} \sum_{n=0}^{L-1} w^2(n)$$

K = number of segments of length L (segments may overlap; if segments are adjacent with no overlap then $N = LK$ where N is the total length of the data record)

(d) $$E[\hat{P}_{xx}(f)] = \frac{1}{K} \sum_{k=0}^{K-1} \left\{ \frac{1}{LU} E\left[\left| \sum_{n=0}^{L-1} w(n)\, x_k(n)\, e^{-j2\pi f n} \right|^2 \right] \right\}$$

since $$E\left[\left| \sum_{n=0}^{L-1} w(n)\, x_k(n)\, e^{-j2\pi f n} \right|^2 \right]$$

$$= \sum_{n=0}^{L-1} \sum_{m=0}^{L-1} w(n)\, w(m)\, E[x_k(n)\, x_k(m)]\, e^{-j2\pi f n} e^{j2\pi f m}$$

$$= \sigma_w^2 \sum_{n=0}^{L-1} w^2(n)$$

and $$L\mathcal{U} = \sum_{n=0}^{L-1} w^2(n)$$

Then

$$E[\hat{P}_{xx}(f)] = \sigma_w^2$$

(e) The use of the window function is to provide sidelobe control (minimize spectral leakage).

$I(f)$ has the best frequency resolution since the Fourier transform is taken over the entire observation record of N data points.

$\hat{P}_{xx}(f)$ has the smallest variance since variance decreases as the number of segments K increases.

(f) $x(n) = A\sin(2\pi f n + \phi)$ and goes through an integer number of cycles in L points.

$$X_k(f) = \sum_{n=0}^{L-1} w(n)\, x_k(n)\, e^{-j2\pi f n}$$

where k is a segment index and the segment length is L.

Recall $$\sin(2\pi f n) = \frac{e^{j2\pi f n} - e^{-j2\pi f n}}{2j}$$

Thus, substituting the expression for $x(n)$

$$|X_k(f)| = \frac{A}{2}\sum_{n=0}^{L-1} w(n) \implies A = \frac{2}{\sum_{n=0}^{L-1} w(n)}\,|X_k(f)|$$

$$\text{sinusoid power} = \frac{A^2}{2} = \frac{2}{\left(\sum_{n=0}^{L-1} w(n)\right)^2}\,|X_k(f)|^2$$

and is the same for all k.

Since $$\hat{P}_{xx}(f) = \frac{1}{K}\sum_{k=0}^{K-1}\frac{1}{L\mathcal{U}}|X_k(f)|^2$$

Then

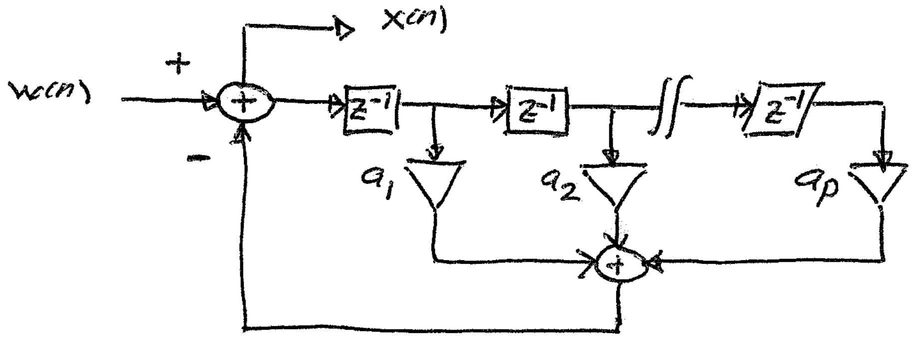
$$\text{sinusoid power} = \frac{A^2}{2} = \frac{2}{\left(\sum_{n=0}^{L-1} w(n)\right)^2}|X_k(f)|^2 = \frac{2}{\left(\sum_{n=0}^{L-1} w(n)\right)^2}\left\{L\mathcal{U}\,\hat{P}_{xx}(f)\right\}$$

Since $U = \frac{1}{L} \sum_{n=0}^{L-1} w^2(n)$

Sinusoid power $= \frac{A^2}{2} = \frac{2}{L} \left\{ \frac{L \sum_{n=0}^{L-1} w^2(n)}{\left(\sum_{n=0}^{L-1} w(n) \right)^2} \right\} \hat{P}_{xx}(f)$

Problem #2

(a)



$$x(n) = w(n) - \sum_{i=1}^{p} a_i x(n-i)$$

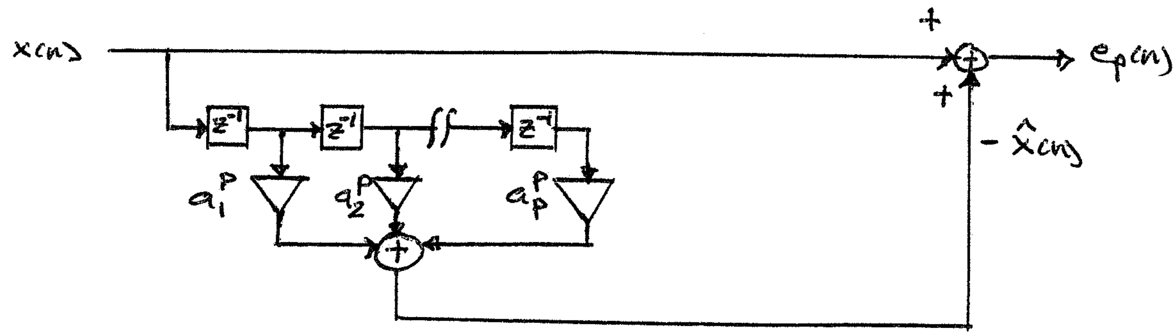
$$H(z) = \frac{1}{\sum_{i=0}^{p} a_i z^{-i}} \qquad a_0 = 1$$

$$= \frac{z^p}{\sum_{i=0}^{p} a_i z^{p-i}} \qquad a_0 = 1$$

(b) $$P_{xx}(f) = \sigma_w^2 \left| H(z) \right|^2_{z = e^{j2\pi f}}$$

$$= \frac{\sigma_w^2}{\left| \sum_{i=0}^{p} a_i e^{-j2\pi f i} \right|^2} \qquad a_0 = 1$$

(c) one-step forward linear predictor of $x(n)$



Problem: $\min\limits_{\underline{a}^p} E[e_p^2(n)]$

Define $\underline{a}^p = \begin{bmatrix} a_1^p \\ a_2^p \\ \vdots \\ a_p^p \end{bmatrix}$ $\quad \underline{x}^-(n) = \begin{bmatrix} x(n-1) \\ x(n-2) \\ \vdots \\ x(n-p) \end{bmatrix}$

$$\phi(m) = E[x(n)x(n+m)] = \phi(-m)$$

$$\underline{g} = E[x(n)\,\underline{x}^-(n)] = \begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(p) \end{bmatrix}$$

$$\underline{\Phi} = E[\underline{x}^-(n)\,\underline{x}^{-T}(n)] = \begin{bmatrix} \phi(0) & \cdots & \phi(p-1) \\ \vdots & \ddots & \vdots \\ \phi(p-1) & \cdots & \phi(0) \end{bmatrix}$$

Since $x(n)$ is wide sense stationary, $\underline{\Phi}$ is Toeplitz and is completely defined by its upper row $\phi(m)$, $m = 0, 1, \ldots, p-1$.
Since $x(n)$ is real, $\underline{\Phi}$ is symmetric, $\underline{\Phi} = \underline{\Phi}^T$.

$$e_p(n) = x(n) + \sum_{i=1}^{p} a_i^p x(n-i)$$

$$= x(n) + \underline{a}^{pT}\underline{x}^-(n)$$

$$e_p^2(n) = \left(x(n) + \underline{a}^{pT}\underline{x}^-(n) \right)^2$$

$$= \left(x(n) + \underline{a}^{pT}\underline{x}^-(n) \right)\left(x(n) + \underline{x}^{-T}(n)\underline{a}^p \right)$$

$$= x^2(n) + 2\,\underline{a}^{pT}x(n)\underline{x}^-(n) + \underline{a}^{pT}\underline{x}^-(n)\underline{x}^{-T}(n)\underline{a}^p$$

$$E[e_p^2(n)] = \sigma_x^2 + 2\,\underline{a}^{pT}\underline{g} + \underline{a}^{pT}\underline{\Phi}\,\underline{a}^p$$

minimizing $E[e_p^2(n)]$ with respect to $\underline{a}^p$ leads to

$$0 = \underline{g} + \underline{\Phi}\,\underline{a}^p \quad \text{or} \quad \underline{\Phi}\,\underline{a}^p = -\underline{g}$$

Solving for $\underline{a}^p$:

$$\underline{a}^p = -\underline{\Phi}^{-1}\underline{g}$$

(d) Forward prediction error power

$$E_p = \min_{\underline{a}^p} E[e_p^2(n)] = \sigma_x^2 + 2\,\underline{a}^{pT}\underline{g} + \underline{a}^{pT}\underline{\Phi}(-\underline{\Phi}^{-1}\underline{g})$$

$$= \sigma_x^2 + \underline{a}^{pT}\underline{g} = \phi(0) + \underline{a}^{pT}\begin{bmatrix}\phi(1)\\ \phi(2)\\ \vdots\\ \phi(p)\end{bmatrix}$$

(e) From the solution in (a), $\underline{\Phi}\,\underline{a}^p = -\underline{g}$

$$\begin{bmatrix} \phi(0) & \cdots & \phi(p-1) \\ \vdots & \ddots & \vdots \\ \phi(p-1) & \cdots & \phi(0) \end{bmatrix} \begin{bmatrix} a_1^p \\ a_2^p \\ \vdots \\ a_p^p \end{bmatrix} = -\begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(p) \end{bmatrix}$$

Adding $[\phi(1)\ \phi(2) \ldots \phi(p)]^T$ to both sides yields

$$\begin{bmatrix} \phi(1) & \phi(0) & \cdots & \phi(p-1) \\ \phi(2) & \phi(1) & & \phi(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(p) & \phi(p-1) & \cdots & \phi(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^p \\ a_2^p \\ \vdots \\ a_p^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Lastly, augmenting the above with the expression for E_p

$$\begin{bmatrix} \phi(0) & \phi(1) & \cdots & \phi(p) \\ \phi(1) & \phi(0) & \cdots & \phi(p-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(p) & \phi(p-1) & \cdots & \phi(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^p \\ \vdots \\ a_p^p \end{bmatrix} = \begin{bmatrix} E_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$