Statistical Signal Processing – April 2012

1. Let x(n) be a real, zero mean, wide sense stationary, discrete random process. One realization of x(n), $0 \leq n \leq N-1$, is observed.

(a) Write the expression for the biased estimate of the autocorrelation, $c_{xx}(m) = \hat{\phi}_{xx}(m)$, where:

$$\phi_{xx}(m) = E[x(n)x(n+|m|)]. \quad \text{(5 points)}$$

(b) Write the expression for the periodogram estimate of the power spectrum, I(f), both in terms of $c_{xx}(m)$ and directly in terms of the Fourier transform of the original observed sequence, x(n). (5 points)

(c) Write the expression for the conventional estimate of the power spectrum, $\hat{P}_{xx}(f)$, in terms of Fourier transforms or DFTs of windowed data segments of length L taken from x(n) (also known as Welch's method of averaging modified periodograms). (10 points)

(d) Assuming x(n) is an uncorrelated random process with variance σ_w^2, calculate the expected value of Welch's method in (c), $E[\hat{P}_{xx}(f)]$. (15 points)

(e) Comment on the reason for using a window function in the calculation of $\hat{P}_{xx}(f)$ and compare (qualitatively) I(f) and $\hat{P}_{xx}(f)$ in terms of frequency resolution and variance (5 points).

2. Consider the following order p autoregressive (AR) process x(n):

$$x(n) = w(n) - \sum_{i=1}^{p} a_i x(n-i)$$

where the a_i are real coefficients and w(n) is a real, wide sense stationary, zero mean, white Gaussian noise sequence with variance σ_w^2.

(a) Provide a block diagram illustrating how x(n) is generated and write the expression for the z-transform of the all-pole filter H(z) driven by w(n). (5 points)

(b) Write the expression for the power spectrum of x(n) in terms of the all-pole filter coefficients a_i and the variance σ_w^2 of w(n). (5 points)

(c) Derive the optimal (in a MMSE sense) one-step forward linear predictor of length p with filter coefficients a_i^p for the AR process x(n) given the autocorrelation sequence $\phi_{xx}(m)$ of x(n). (30 points)

(d) Substitute the solution for the optimal a_i^p back into the expression for forward prediction error power obtained in (c) thus providing an expression for the minimum forward prediction error power, E_p, in terms of the a_i^p and $\phi_{xx}(m)$. (10 points)

(e) Augment the solution in (c) with the expression for minimum forward prediction error power in (d) to yield an expression involving the data autocorrelation matrix, Φ, the one-step forward prediction error filter, and the minimum forward prediction error power, E_p. (10 points)