Statistical Signal Processing – ECE 251A - October 2012

1. Let x(n) be a real, zero mean, wide sense stationary, discrete random process. Given the definition for the auto and cross-correlation functions:

$$\phi_{xx}(m) = E[x(n)x(n+m)]$$

$$\phi_{xy}(m) = E[x(n)y(n+m)]$$

and the expression for the output y(n) of a discrete-time linear system in terms of its input x(n) and its shift-invariant impulse response h(n):

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

(a) Express the autocorrelation of the output $\phi_{yy}(m)$ and the cross-correlation of the input and the output $\phi_{xy}(m)$ in terms of the input autocorrelation $\phi_{xx}(m)$ and the impulse response of the system h(n). (20 points)

(b) Specialize the results in (a) to the case when x(n) is an uncorrelated random process. (10 points)

2. Let x(n) be a real, zero mean, wide sense stationary, discrete random process. One realization of x(n), $0 \le n \le N-1$, is observed.

(a) Write the expression for the conventional estimate of the power spectrum, $\hat{P}_{xx}(f)$, in terms of Fourier transforms or DFTs of windowed data segments of length L taken from x(n) (also known as Welch's method of averaging modified periodograms). (5 points)

(b) Assuming x(n) is an uncorrelated random process with variance σ_w^2, calculate the expected value of Welch's method in (b), $E[\hat{P}_{xx}(f)]$. (10 points)

(c) Comment on the reason for using a window function in the calculation of $\hat{P}_{xx}(f)$ and compare (qualitatively) the impact of averaging on $\hat{P}_{xx}(f)$ in terms of frequency resolution and variance. (5 points)

(d) If x(n) is a sinusoid instead of a discrete random process, x(n) = A sin (2πfn + ø), explain how to recover the power of the sinusoid ($A^2/2$) from $\hat{P}_{xx}(f)$ (assume f is at a DFT bin center). (10 points)

3. Consider the following order p autoregressive (AR) process x(n):

$$x(n) = w(n) - \sum_{i=1}^{p} a_i x(n-i)$$

where the a_i are real coefficients and w(n) is a real, wide sense stationary, zero mean, white Gaussian noise sequence with variance σ_w^2.

(a) Derive the optimal (in a MMSE sense) one-step forward linear predictor of length p with filter coefficients a_i^p for the AR process x(n) given the autocorrelation sequence $\phi_{xx}(m)$ of x(n). (30 points)

(b) Substitute the solution for the optimal a_i^p back into the expression for forward prediction error power obtained in (a) thus providing an expression for the minimum forward prediction error power, E_p, in terms of the a_i^p and $\phi_{xx}(m)$. (5 points)

(c) Augment the solution in (a) with the expression for minimum forward prediction error power in (b) to yield an expression involving the data autocorrelation matrix, Φ, the one-step forward prediction error filter, and the minimum forward prediction error power, E_p. (5 points)