Math Question-Solutions

Part 1

(i) Solve the following set of coupled differential equations

$$\begin{aligned} \frac{dA(z)}{dz} &= j(\gamma/2)\, B(z) \\ \frac{dB(z)}{dz} &= j(\gamma/2)\, A(z) \end{aligned}$$

subject to the boundary conditions that $A(0) = C$ and $B(0) = 0$. (Note that $j \doteq \sqrt{-1}$)

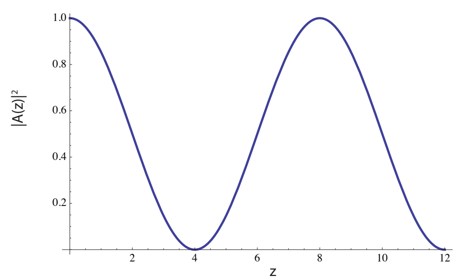
(ii) Sketch the solutions for $|A(z)|^2$ and $|B(z)|^2$ for a value of $\gamma = \pi/4$.

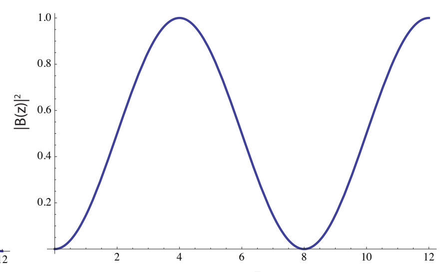
Solution

Taking the derivative of each side and back substituting we obtain

$$\frac{d^2A(z)}{dz^2} + \left(\frac{\gamma}{2}\right)^2 A(z) = 0.$$

Using the boundary condition, this has solution $A(z) = C\cos(\gamma z/2)$. Back solving we obtain $B(z) = jC\sin(\gamma z/2)$. The plots are below for $C = 1$ and $\gamma = \pi/4$.





Part 2

A zero-mean joint Gaussian probability distribution is given by

$$f_{\underline{xy}}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right).$$

(i) Derive the probability distribution $f_{\underline{z}}(z)$ for the random variable

$$\underline{z} \doteq \underline{x}^2 + \underline{y}^2,$$

which is the squared-magnitude of the joint Gaussian probability distribution and sketch this distribution for $\sigma^2 = 1$.

Solution

Using a cartesian-polar coordinate transformation given by $x = r\cos\phi$ and $y = r\sin\phi$ the joint-gaussian distribution in polar coordinates is

$$f_{\underline{r\phi}}(r,\phi) = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right),$$

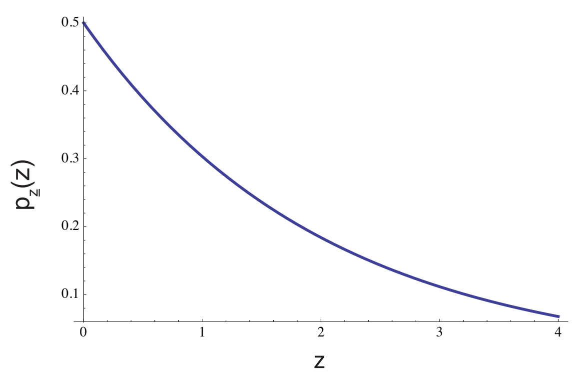
where $\sin^2\phi + \cos^2\phi = 1$ and the factor of r is from the differential area relationship $dxdy = rdrd\phi$. Now set $r^2 = z$ so that $dz = 2rdr$. The new joint pdf $f_{\underline{z\phi}}(z,\phi)$ is

$$f_{\underline{z\phi}}(z,\phi) = \frac{1}{4\pi\sigma^2}\exp\left(-\frac{z}{2\sigma^2}\right)dzd\phi.$$

Integrating over ϕ, the marginal distribution for the squared-magnitude $z = x^2 + y^2$ is

$$\begin{aligned} f_{\underline{z}}(z) &= \int_0^{2\pi}\frac{1}{4\pi\sigma^2}\exp\left(-\frac{z}{2\sigma^2}\right)d\phi \\ &= \frac{1}{2\sigma^2}e^{-z/2\sigma^2} \end{aligned}$$

which is an exponential distribution with a mean value of $2\sigma^2$. A plot is shown below



(ii) Derive the probability distribution $f_{\underline{\phi}}(\phi)$ for the random variable

$$\underline{\phi} \doteq \tan^{-1}\left(\underline{y}/\underline{x}\right),$$

which is the phase of the joint Gaussian distribution and sketch this distribution for $\sigma^2 = 1$.

Solution

The marginal phase distribution is determined by integrating $f_{\underline{z\phi}}(z,\phi)$ over z. This yields

$$f_{\underline{\phi}}(\phi) = \frac{1}{2\pi}$$

which is a unform distribution.