Signals and Systems problem for the Spring 2013 MS Exam in ECE

Suppose a causal, linear, time-invariant system with input $x(t)$ and output $y(t)$ is described by the differential equation

$$\frac{d^2y}{dt^2} - \frac{dx}{dt} + 5\frac{dy}{dt} = x - 6y$$

Find the output $y(t)$ when the input is $x(t) = 2\delta(t-1) - 4e^{-4t+4}u(t-1)$, where δ is the Dirac delta function and u is the unit step function.

SOLUTION: First, take Laplace transforms to find the system function $H(s)$.

$$\begin{aligned}
\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y &= \frac{dx}{dt} + x \\
s^2Y(s) + 5sY(s) + 6Y(s) &= sX(s) + X(s) \\
H(s) &= \frac{Y(s)}{X(s)} = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)(s+3)}
\end{aligned}$$

Let $\hat{x}(t) = \delta(t) - 2e^{-4t}u(t)$. Then $x(t) = 2\hat{x}(t-1)$. We find the output $\hat{y}(t)$ if $\hat{x}(t)$ is the input and then use LTI properties to deduce the output due to $x(t)$.

$$\begin{aligned}
&\hat{X}(s) = 1 - \frac{2}{s+4} = \frac{s+2}{s+4} \\
&\hat{Y}(s) = H(s)\hat{X}(s) = \frac{s+1}{(s+2)(s+3)} \cdot \frac{s+2}{s+4} = \frac{s+1}{(s+3)(s+4)} = \frac{3}{s+4} - \frac{2}{s+3} \\
&\therefore \hat{y}(t) = 3e^{-4t}u(t) - 2e^{-3t}u(t) \\
&\therefore y(t) = 2\hat{y}(t-1) = 6e^{-4(t-1)}u(t-1) - 4e^{-3(t-1)}u(t-1)
\end{aligned}$$