Math Question Solutions

Part 1

(i) Find the eigenvalues and eigenvectors of the following matrix

$$M = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

Solution
The eigenvalues are

$$\begin{aligned} \det \begin{bmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{bmatrix} &= \lambda^2 - \lambda - 2 = 0 \\ &= (\lambda - 2)(\lambda + 1) \end{aligned}$$

so that the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$.

The eigenvector v_1 is given by

$$\begin{bmatrix} 3-\lambda_1 & -1 \\ 4 & -2-\lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} \to x_1 - y_1 = 0 \to v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvector v_2 is given by

$$\begin{bmatrix} 3-\lambda_2 & -1 \\ 4 & -2-\lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix} \to 4x_2 - y_2 = 0 \to v_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

(ii) Using a linear combination of the eigenvectors, construct the basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution

For $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

or $a + b = 1$ and $a + 4b = 0$. Solving we have $a = 4/3$ and $b = -1/3$

For $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we have

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

or $c + d = 0$ and $c + 4d = 1$. Solving we have $c = -1/3$ and $d = 4/3$.

(iii) If one number is changed in the matrix, then the corresponding eigenvectors will be orthogonal. Determine this number and verify that the resulting eigenvectors are orthogonal.

Solution

Orthogonal eigenvectors can be constructed if the matrix is symmetric. Therefore, there are two solutions. If the off-diagonal elements each have a value of 4 then the corresponding eigenvectors are

$$\begin{bmatrix} \frac{1}{2}+\frac{1}{8}\left(1+\sqrt{89}\right) \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{1}{2}+\frac{1}{8}\left(1-\sqrt{89}\right) \\ 1 \end{bmatrix}$$

and are orthogonal because the inner product is zero

$$\left(\frac{1}{2}+\frac{1}{8}\left(1+\sqrt{89}\right)\right)\left(\frac{1}{2}+\frac{1}{8}\left(1-\sqrt{89}\right)\right)+1=0.$$

If the off-diagonal elements each have a value of -1, then the corresponding eigenvectors are

$$\begin{bmatrix} \frac{1}{2}\left(1+\sqrt{29}\right) \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{1}{2}\left(1-\sqrt{29}\right) \\ 1 \end{bmatrix}.$$

Part 2 A joint probability distribution $p_{XY}(x,y)$ is given by

$$p_{XY}(x,y) = A(8x+14yx-21y-12)$$

where $0 < x < 1/2$, $-1 < y < -1/2$, and A is a constant.

i) Find A

Solution
The pdf must integrate to unity so

$$\begin{aligned}
\int_{-1}^{-1/2}\int_{0}^{1/2} p_{XY}(r,\theta)dxdy &= 1\\
\int_{-1}^{-1/2}\int_{0}^{1/2} p_{XY}(r,\theta)dxdy &= 1\\
A\int_{-1}^{-1/2}\int_{0}^{1/2} (2x-3)(7y+4)dxdy &= 1\\
A\left(\left[x^2-3x\Big|_0^{1/2}\right]\int_{-1}^{-1/2}(7y+4)dy\right) &= 1\\
A\left(-\frac{5}{4}\int_{-1}^{-1/2}(7y+4)dy\right) &= 1\\
A\left(-\frac{5}{4}\left[\frac{7y^2}{2}+4y\Big|_{-1}^{-1/2}\right]\right) &= 1\\
A\left(-\frac{5}{4}\right)\left(-\frac{5}{8}\right) &= 1\\
A &= \frac{32}{25}.
\end{aligned}$$

ii) Find the marginal distributions for $p_X(x)$ and $p_Y(y)$

Solution
The marginal distribution is obtained by integrating out y in the joint pdf

$$\begin{aligned}
p_X(x) &= \int_{-1}^{-1/2} p_{XY}(x,y)dy \\
&= \frac{32}{25}\int_{-1}^{-1/2}(2x-3)(7y+4)dy \\
&= \left(\frac{32}{25}\right)\left(-\frac{5}{8}\right)(2x-3) \\
&= \frac{4}{5}(3-2x)
\end{aligned}$$

Similarly,

$$\begin{aligned}
p_Y(y) &= \int_{0}^{1/2} p_{XY}(x,y)dx \\
&= \frac{32}{25}\int_{0}^{1/2}(2x-3)(7y+4)dy \\
&= \left(\frac{32}{25}\right)\left(-\frac{5}{4}\right)(7y+4) \\
&= -\frac{8}{5}(7y+4)
\end{aligned}$$

iii) Are the random variables defined by X and Y independent? (Justify your answer.)

Solution
The joint pdf can be factored into

$$p_{XY}(x,y) = \frac{32}{25}(8x+14yx-21y-12) = \frac{32}{25}(2x-3)(7y+4)$$

Since the joint pdf can be factored into a product of marginal pdfs, the random variables X and Y are independent.