Math Question

(You can attempt all three parts. A passing grade is at least 50% for two out of the three parts.)

Part 1

Find the solution of the initial value problem

$$\frac{dy}{dt} - 2ty = 1 \qquad y(0) = 1$$

Note: $\int_0^t e^{-\tau^2} d\tau = (\sqrt{\pi}/2)\text{erf(t)}$ where erf(t) is the error function.

Solution

This is a first order-differential equation with a time-varying coefficient $-2t$. To solve this equation, we first determine the integrating factor $\mu(t)$, which is given by

$$u(t) = \exp\left(-\int^t 2t' dt'\right) = e^{-t^2}.$$

The integrating factor is defined such that

$$\frac{du}{dt}\frac{1}{u} = -2t.$$

Substitute this expression into the original equation and multiply both sides by $u(t)$. The left side is then $u\, dy/dt + y\, du/dt = d(u\, y)/dt$ so that

$$\frac{d}{dt}\left(u(t)y(t)\right) = u(t).$$

Integrating both sides and dividing through by $u(t)$ yields

$$y(t) = e^{t^2}\left[\int_0^t e^{-t'^2} dt' + c\right].$$

Applying the initial condition $y(0) = 1$, yields $c = 1$. Using the integral given in the problem statement, the final solution is

$$y(t) = e^{t^2}\left[\frac{\sqrt{\pi}}{2}\text{erf}(t) + 1\right].$$

Part 2

Find the general solution of the following matrix equation

$$\begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \end{bmatrix}$$

Solution

This is a inhomogeneous matrix equation. The complete solution consists of the sum of a particular solution and a homogeneous solution. From inspection, the particular solution is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The homogeneous solution is a vector

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

that satisfies the following set of homogeneous equations

$$\begin{aligned} 6y_1 + 5y_2 + 4y_3 &= 0 \\ 3y_1 + 2y_2 + y_3 &= 0 \end{aligned}$$

The complete solution is the sum of the homogeneous part and the inhomogeneous part

$$\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Part 3

A uniform random variable X is defined over the interval $[0, 2]$. A second independent random variable Y has a probability density function $P_Y(y) = 2e^{-2y}u(y)$ where $u(y)$ is a unit-step function. Find and sketch the probability density function $P_Z(z)$ for the random variable $Z = X + Y$.

Solution

The probability density function of the sum of two independent random variables is the convolution of the individual probability density functions. The pdf for a uniform random variable is a rect function with a height that is equal to the reciprocal of the interval. For an interval of two, the height is then 1/2. Evaluating the convolution, there are two intervals. For $z < 2$, we have

$$P_Z(z) = \int_0^z e^{-2z'} dz' = \frac{1}{2}\left(1 - e^{-2z}\right) \qquad z < 2$$

where the factor of 2 for $P_Y(y)$ cancels the factor of 1/2 for the rect function. For $z \geq 2$, we have

$$P_Z(z) = \int_{z-2}^z e^{-2z'} dz' = \frac{1}{2} e^{-2z}\left(e^4 - 1\right) \qquad z \geq 2$$

A plot is shown below