Problem 1:

a) \[ \hat{R}(e^{j\omega}) = \frac{1}{KU} \sum_{i=0}^{K-1} \sum_{n=0}^{L-1} x[n + iD]w[n]e^{j\omega n} \]

b) \( K \): number of segments. more segments \( \Rightarrow \) lower variance

\( L \): length of segment, longer segment results in higher resolution

\( w[n] \): shapes the bandpass filter. Tradeoff between mainlobe width and sidelobe level

\( D \): Decides overlap between segment. \( D \) new samples per segment. Overlaps = \( L-D \). More overlap implies more segments but correlated segments and so reduction is not only dependent on \( K \) but also on \( D \).

\( U \): Normalizing constant to make the estimate unbiased \( U = \frac{1}{L} \sum_{n=0}^{L-1} w^2[n] \).

c) \( U \) is chosen to make the estimate unbiased. The basic steps for deriving this are as follows:

\[ E(\hat{R}(e^{j\omega})) = \frac{1}{KU} \sum_{i=0}^{K-1} \frac{1}{L} E(|\sum_{n=0}^{L-1} x[n + iD]w[n]e^{j\omega n}|^2) = \frac{1}{U} \frac{1}{L} E(|\sum_{n=0}^{L-1} x[n]w[n]e^{j\omega n}|^2) \]

This simplification is possible by using the WSS property of the sequence making the expectation independent of the segment index \( i \). Now using a filterbank interpretation it can be shown that (See your class notes)

\[ E(|\sum_{n=0}^{L-1} x[n]w[n]e^{j\omega n}|^2) \approx \sum_{n=0}^{L-1} |w[n]|^2 R(e^{j\omega}) \]

Substituting this in the above equation we have

\[ E(\hat{R}(e^{j\omega})) \approx \frac{1}{U} \frac{1}{L} \sum_{n=0}^{L-1} |w[n]|^2 R(e^{j\omega}) \]

The condition of unbiasedness requires \( E(\hat{R}(e^{j\omega})) = R(e^{j\omega}) \) leading to the choice of \( U = \frac{1}{L} \sum_{n=0}^{L-1} w^2[n] \).
Problem 2:

a) $H(z) = 2 + 3z^{-1}$. Hence the power spectrum of $x[n]$ is

$$R_x(e^{j\omega}) = \sigma_x^2 |H(e^{j\omega})|^2 = |2 + 3e^{-j\omega}|^2 = (2 + 3e^{-j\omega}) (2 + 3e^{j\omega}) = 6e^{j\omega} + 13 + 6e^{-j\omega}$$

Hence the autocorrelation sequence is

$$r[0] = 13, r[1] = r[-1] = 6, r[m] = r[-m] = 0, m \geq 2$$

b) The process is a MA process since the auto-correlation sequence is of finite duration. The Blackman-Tukey approach can be used for estimating the power spectrum.

$$R_{BT}(e^{j\omega}) = \sum_{m=-L}^{L} \hat{r}_{xx}[m] w[m] e^{-j\omega m}$$

$\hat{r}_{xx}[m]$ is an estimate of the autocorrelation sequence. If order of MA process $Q$ is assumed unknown, then choose $L = Q = 1$ and a rectangular window $w[m] = 1$ can be used.

c) $H_{min}(z) = 3 + 2z^{-1}$

d) Yule Walker equation:

$$\begin{bmatrix} r[0] & r[-1] \\ r[1] & r[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} r[1] \\ r[2] \end{bmatrix}$$

or

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} 13 & 6 \\ 6 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = - \frac{1}{133} \begin{bmatrix} 13 & -6 \\ -6 & 13 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = - \frac{1}{133} \begin{bmatrix} 78 \\ -36 \end{bmatrix}$$

$$\sigma_x^2 = r[0] + a_1 r[1] + a_2 r[2] = 13 + \frac{78}{133} + 0 = 9.48$$

AR Power Spectrum is given by:

$$R(e^{j\omega}) = \sigma_x^2 \left| \frac{1}{1 + a_1 e^{-j\omega} + a_2 e^{-2j\omega}} \right|^2$$