1. During transmission a random process, \(X(t)\), is corrupted by additive white noise, \(N(t)\). The processes \(X(t)\) and \(N(t)\) are independent, wide sense stationary and both have zero mean. The correlation function of \(X(t)\) is

\[ R_X(\tau) = e^{-|\tau|}. \]

The power spectral density of the white noise is \(P_0 = 1/5\). The sum \(X(t) + N(t)\) is passed through an ideal low-pass filter with one-sided bandwidth \(\Omega\). That is

\[ H(i\omega) = \begin{cases} 1, & -\Omega \leq \omega \leq \Omega \\ 0, & \text{otherwise.} \end{cases} \]

Determine the bandwidth \(\Omega\) that minimizes the error

\[ \varepsilon = E \left[ (X(t) - \hat{X}(t))^2 \right]. \]

[NOTE: This is not a problem to determine the optimum filter, but only to find the optimum bandwidth \(\Omega\).]

Optimum Bandwidth =
Some Useful Formulas

\[ \Phi_x(u) = \int_{-\infty}^{\infty} f_x(x)e^{iux} \, dx \]

\[ f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(u)e^{-iux} \, du \]

\[ S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-i\omega\tau} \, d\tau \]

\[ R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)e^{i\omega\tau} \, d\omega \]

\[ \sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}, |\rho| < 1 \]

\[ \cos A \cos B = \frac{1}{2} \cos(A - B) + \frac{1}{2} \cos(A + B) \]

\[ \sin A \sin B = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B) \]

\[ \int_{\delta(x)} e^{iux} \, dx = 1 \]

\[ \int_{0}^{\infty} \left\{ \alpha^{n+1} x^n e^{-\alpha x} \right\} e^{iux} \, dx = \left( \frac{\alpha}{\alpha - iu} \right)^{n+1}, \quad \alpha > 0 ; \, n = 0, 1, \ldots \]

\[ \int_{-\infty}^{\infty} \left\{ \frac{\alpha}{\pi(\alpha^2 + x^2)} \right\} e^{iux} \, dx = e^{-\alpha|u|}, \quad \alpha > 0 \]

\[ \int_{-\infty}^{\infty} \left\{ \frac{1}{2\alpha} \right\} e^{iux} \, dx = \frac{\sin \alpha u}{\alpha u}, \quad \alpha > 0 \]