1. Let $x(n)$ be a real, zero mean, wide sense stationary, discrete random process. Given the definition for the auto and cross-correlation functions:

\[
\phi_{xx}(m) = E[x(n)x(n+m)] \\
\phi_{xy}(m) = E[x(n)y(n+m)]
\]

and the expression for the output $y(n)$ of a discrete-time linear system in terms of its input $x(n)$ and its shift-invariant impulse response $h(n)$:

\[
y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)
\]

(a) Express the autocorrelation of the output $\phi_{yy}(m)$ and the cross-correlation of the input and the output $\phi_{xy}(m)$ in terms of the input autocorrelation $\phi_{xx}(m)$ and the impulse response of the system $h(n)$. (20 points)

(b) Specialize the results in (a) to the case when $x(n)$ is an uncorrelated random process. (10 points)

2. Let $x(n)$ be a real, zero mean, wide sense stationary, discrete random process. One realization of $x(n)$, $0 \leq n \leq N-1$, is observed.

(a) Write the expression for the conventional estimate of the power spectrum, $\hat{P}_{xx}(f)$, in terms of Fourier transforms or DFTs of windowed data segments of length $L$ taken from $x(n)$ (also known as Welch’s method of averaging modified periodograms). (5 points)

(b) Assuming $x(n)$ is an uncorrelated random process with variance $\sigma_w^2$, calculate the expected value of Welch’s method in (b), $E[\hat{P}_{xx}(f)]$. (10 points)

(c) Comment on the reason for using a window function in the calculation of $\hat{P}_{xx}(f)$ and compare (qualitatively) the impact of averaging on $\hat{P}_{xx}(f)$ in terms of frequency resolution and variance. (5 points)

(d) If $x(n)$ is a sinusoid instead of a discrete random process, $x(n) = A \sin (2\pi fn + \phi)$, explain how to recover the power of the sinusoid ($A^2/2$) from $\hat{P}_{xx}(f)$ (assume $f$ is at a DFT bin center). (10 points)
3. Consider the following order $p$ autoregressive (AR) process $x(n)$:

$$x(n) = w(n) - \sum_{i=1}^{p} a_i x(n - i)$$

where the $a_i$ are real coefficients and $w(n)$ is a real, wide sense stationary, zero mean, white Gaussian noise sequence with variance $\sigma_w^2$.

(a) Derive the optimal (in a MMSE sense) one-step forward linear predictor of length $p$ with filter coefficients $a_i^p$ for the AR process $x(n)$ given the autocorrelation sequence $\phi_{xx}(m)$ of $x(n)$. (30 points)

(b) Substitute the solution for the optimal $a_i^p$ back into the expression for forward prediction error power obtained in (a) thus providing an expression for the minimum forward prediction error power, $E_p$, in terms of the $a_i^p$ and $\phi_{xx}(m)$. (5 points)

(c) Augment the solution in (a) with the expression for minimum forward prediction error power in (b) to yield an expression involving the data autocorrelation matrix, $\Phi$, the one-step forward prediction error filter, and the minimum forward prediction error power, $E_p$. (5 points)